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Original Research Paper

Grouping of Terms of a Series

Corresponding Author:
Suman Gautam
Mahendra Multiple Campus, Nepalgunj,Tribhuvan University, Nepal

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ABSTRACT:
From a given series we can make a new series by inserting brackets (parentheses) among the terms of the given series that is, making the groups of terms of the series. The terms of the new series are formed by summing the terms of the series within the inserted brackets. If Σy_n is a series obtained from a series Σx_n by inserting parentheses'(…..)' then Σx_n is also called a series obtained from the series Σy_n by removing parentheses.

Keywords: Grouping, brackets, parentheses, summing, inserting, removing, converges/convergent, diverges/divergent

INTRODUCTION:

Let us Consider A Series:

$\Sigma x_n = x_1 + x_2 + x_3 + x_4 + x_5 + \ldots\ldots\ldots\ldots + x_n + \ldots\ldots\ldots$

Let us make groups of terms of this series by inserting brackets in it as follows,

$\Sigma x_n = (x_1+x_2+\ldots\ldots\ldots+x_{10}) + (x_{11}+x_{12}+\ldots\ldots\ldots x_{20}) + (x_{21}+x_{22}+\ldots\ldots\ldots x_{30}) + \ldots\ldots\ldots\ldots\ldots$

(Where each group may contain different numbers of terms)

Now denote the subscripts 10=k(1), 11=k(1)+1,……,20=k(2), 21=k(2)+1,………., 30=k(3), and so on. Then we have-

$\Sigma x_n = (x_1+x_2+\ldots\ldots\ldots+x_{10}) + (x_{11}+x_{12}+\ldots\ldots\ldots x_{20}) + (x_{21}+x_{22}+\ldots\ldots\ldots x_{30}) + \ldots\ldots\ldots\ldots\ldots$

$= (x_1+x_2+\ldots\ldots\ldots+x_{k(1)}) + (x_{k(1)+1} +x_{k(1)+2}+\ldots\ldots\ldots x_{k(2)}) + (x_{k(2)+1}+x_{k(2)+2}+\ldots\ldots\ldots x_{k(3)}) + \ldots\ldots\ldots\ldots\ldots$

Let, $y_1= x_1+x_2+\ldots\ldots\ldots+x_{10} = x_1+x_2+\ldots\ldots\ldots+x_{k(1)}$

$Y_2= x_{11}+x_{12}+\ldots\ldots\ldots x_{20} = x_{k(1)+1} +x_{k(1)+2}+\ldots\ldots\ldots x_{k(2)}$

$Y_3 = x_{21}+x_{22}+\ldots\ldots\ldots x_{30} = x_{k(2)+1}+x_{k(2)+2}+\ldots\ldots\ldots x_{k(3)}$

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$Y_{n+1} = x_{k(n)+1}+x_{k(n)+2}+\ldots\ldots\ldots x_{k(n+1)}$

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Thus we Obtain a New Series:

$\Sigma y_n = y_1 + y_2 + y_3 + y_4 + y_5 + \ldots\ldots\ldots+ y_n + y_{n+1} + \ldots\ldots\ldots$

In this case, the new series Σy_n is called a series obtained from the series Σx_n by grouping terms or inserting brackets (parentheses). The series Σx_n is called a series obtained from the series Σy_n by ungrouping terms or removing brackets (parentheses). For example,

Let, $\Sigma x_n= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots\ldots\ldots$,and-

Let, $\Sigma y_n = (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots\ldots\ldots\ldots = 0 + 0 + 0 + 0 + 0 + \ldots\ldots\ldots\ldots\ldots$

Then, Σy_n is called a series obtained from the series Σx_n by grouping of its terms. Clearly, here the series Σy_n is convergent whereas the series Σx_n is non-convergent (It is oscillating). The formal definition of grouping of terms of series is as follows,

DEFINITION:

Let $\{k_{(n)}\}$ denotes a strictly increasing sequence of positive integers, i.e., $k_{(1)}<k_{(2)}<k_{(3)}<\ldots\ldots<k_{(n)}<\ldots\ldots$, and $k_{(n)}\in Z^{+}$. Let Σx_n and Σy_n be two series related as follows,

$y_1= x_1+x_2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x_{k(1)}$

$Y_{n+1}= x_{k(n)+1}+x_{k(n)+2}+\ldots\ldots\ldots\ldots\ldots+x_{k(n+1)}$ for n=1, 2, 3, 4, ………(i.e. , $n\in Z^{+}$)

Then the series Σy_n is said to be obtained from the series Σx_n by grouping of its terms or inserting brackets(parentheses).The series Σx_n is said to be obtained from the series Σy_n by ungrouping of its terms or removing brackets(parentheses).

Some Theorems:

Theorem-1:

If a series Σx_n converges to s then every series Σy_n obtained from Σx_n by grouping of its terms also converges to s.

Proof

Let $\Sigma x_n= s$ and let Σy_n is a series obtained from Σx_n by grouping of its terms as follows,

$y_1= x_1+x_2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x_{k(1)}$

$Y_{n+1}= x_{k(n)+1}+x_{k(n)+2}+\ldots\ldots\ldots\ldots\ldots+x_{k(n+1)},$ (where $k_{(n)}<k_{(n+1)}$ and n=1,2,3,4,………)

We show: $\Sigma y_n= s$.

For this, let $\{s_n\}$ and $\{t_n\}$ be the sequences of partial sums of the series Σx_n and Σy_n respectively. Then as $\Sigma x_n= s$, so

$\lim_{n\to\infty} s_n= s$. Now, $t_n= y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_n = x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots + X_{k(n)} = S_{k(n)}$.

$\Rightarrow$ $\{t_n\}$ is a subsequence of $\{s_n\}$, as $\{s_{k(n)}\}$ is a subsequence of $\{s_n\}$.

$\Rightarrow$ $\{t_n\}$ converges to s, as $\{s_n\}$ converges to s.

$\Rightarrow$ Σy_n converges to s, i.e. , $\Sigma y_n = s$. Proved.

Notes:-1- The converse of this theorem may not be true, that is, removing the brackets may destroy the convergence.

For example: - Let $\Sigma x_n = \Sigma(-1)^{n+1}$, $\forall n\in Z^{+}$

That is, Let $\Sigma x_n = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots\ldots\ldots$

Let Σy_n be the series such that k(n) = 2n, then $\Sigma y_n = (1 - 1) + (1 - 1) + (1 - 1) + \ldots\ldots\ldots\ldots\ldots = 0 + 0 + 0 + \ldots\ldots\ldots\ldots\ldots$

Clearly, Σx_n is a series obtained from the series Σy_n by removing brackets. Here, $\Sigma y_n= 0$, i.e. , Σy_n converges to 0, but the series Σx_n do not converge.

2- The converse of the theorem is true if we restrict Σx_n and k as in the following theorem-2.

Theorem-2:

Let Σx_n and Σy_n be two series related as in the above definition. Assume that $\exists$ M>0 such that

$k(n+1) - k(n) < M \quad \forall\, n\in Z^{+}$. Also assume that $\lim_{n\to\infty} x_n = 0$. Then-

Σx_n converges $\Leftrightarrow$ Σy_n converges. Infact, $\Sigma x_n= s \Leftrightarrow \Sigma y_n = s$

Proof

Let the hypothesis of the theorem.

We show: $\Sigma x_n = s \Leftrightarrow \Sigma y_n = s$.

First we show ($\Rightarrow$): For this, Let $\Sigma x_n = s$, then as proceeding the above theorem-1, we obtain $\Sigma y_n = s$.

Next we show ($\Leftarrow$): For this, Let $\Sigma y_n = s$. Further let $\{s_n\}$ and $\{t_n\}$ be the sequences of partial sums of the series Σx_n and Σy_n respectively, then $t_n\to s$ since $\Sigma y_n = s$.

We have to show $\Sigma x_n = \lim_{n\to\infty} s_n = s$.

For this, let $\in > 0$ *be given*.

Since $\lim_{n\to\infty} t_n = s$ and $\lim_{n\to\infty} x_n = 0$ so corresponding to the $\in > 0$, $\exists$ $N\in Z^{+}$ such that-

$\forall n \ge N \Rightarrow |t_n - s| < \in/2 \text{ and } |x_n - 0| = |x_n| < \in/2M$

Now, if n>k(N), we can find m$\geq$N so that N$\leq k(m) \leq n < k(m+1)$. Clearly for such n, we have-

$s_n = x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots + x_n$

$= x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots + x_n + x_{n+1} + x_{n+2} + \ldots\ldots\ldots + x_{k(m+1)} - [x_{n+1} + x_{n+2} + \ldots\ldots\ldots + x_{k(m+1)}]$

$= s_{k(m+1)} - [x_{n+1} + x_{n+2} + \ldots\ldots\ldots + x_{k(m+1)}]$

$= t_{m+1} - [x_{n+1} + x_{n+2} + \ldots\ldots\ldots + x_{k(m+1)}]$

Now, $s_n - s = [t_{m+1} - s] - [x_{n+1} + x_{n+2} + \ldots\ldots\ldots + x_{k(m+1)}]$

$$\Rightarrow |s_n - s| = \left|[t_{m+1} - s] - [x_{n+1} + x_{n+2} + \cdots \ldots\ldots + x_{k(m+1)}]\right|$$

$$\leq |t_{m+1} - s| + \left|x_{n+1} + x_{n+2} + \cdots \ldots\ldots + x_{k(m+1)}\right|$$

$$\leq |t_{m+1} - s| + |x_{n+1}| + |x_{n+2}| + \cdots \ldots\ldots\ldots + \left|x_{k(m+1)}\right|$$

$$\leq |t_{m+1} - s| + \left|x_{k(m)+1}\right| + \left|x_{k(m)+2}\right| + \cdots \ldots\ldots\ldots + \left|x_{k(m+1)}\right|$$

$$(\because x_n \in \{x_{k(m)}, x_{k(m)+1}, x_{k(m)+2}, \ldots\ldots\ldots\ldots, x_{k(m+1)}\})$$

$$\Rightarrow |s_n - s| < \in/2 + [k(m+1) - k(m)].\in/2M < \in/2 + M.\in/2M = \in/2 + \in/2 = \in \quad (\forall n \geq N)$$

Thus we have, corresponding to the $\in > 0$ $\exists$ N$\in Z^+$ such that $\forall n \geq N \Rightarrow |s_n - s| < \in$.

Hence $\lim_{n \to \infty} s_n = s$ and so $\Sigma x_n = s$. Proved.

Concluding Remark:

From above we know that by grouping the terms of a given series by different ways different new series can be formed. Such new series must be convergent if the given series is convergent and the new series may or may not convergent if the given series is divergent.

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